Solvability of the Cauchy problem for multidimensional difference equations in rational cone

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The Cauchy problem for multidimensional difference equations whose solutions are sought at the intersection of rational cone with integer lattice is formulated and sufficient condition for its solvability is given.

Let us assume that $a^1,\dots,a^n$ are *linearly independent* vectors with integer coordinates $a^j=\left(a_1^j,\dots,a_n^j\right)$, $a_i^j\in\mathbb{Z}$. *Rational cone* generated by the vectors $a^1,\dots,a^n$, is the set $K=\{x\in\mathbb{R}^n : x=\lambda_1 a^1+\dots+\lambda_n a^n, \lambda_j\in\mathbb{R}_+, j=1,\dots,n\}$. Let us note that this cone is *simplicial* cone, that is, each element of the cone is uniquely expressed in terms of generators. In addition, simplicial cone is also salient cone, that is, this cone contains no lines. Let us introduce matrix A. The columns of this matrix are composed of the vectors a^j and $\Delta=\det A$. If $\Delta=1$ then the cone K is a unimodular cone.

Let us define a partial order between points $u,v\in\mathbb{R}^n$ as follows $u\underset{K}{\geqslant} v \Leftrightarrow u\in v+K$, where $v+K$ is the shift of the cone K by the vector v. In addition, we write $u\underset{K}{\ngeqslant} v$ if $u\in K\backslash\{v+K\}$.

Let us fix $m\in\Omega$ and formulate the Cauchy problem as follows: Let us find a function $f(x)$ that satisfies the equation

$$P(\delta)f(x)=g(x), x\in K\cap\mathbb{Z}^n \tag{1}$$

and coincides with the given function $\varphi(x)$ on set X_0

$$f(x)=\varphi(x), x\in X_0, \tag{2}$$

where $X_0=\{x\in K\cap\mathbb{Z}^n : x\underset{K}{\ngeqslant} m\}$.

In the positive octant $\mathbb{Z}_+^n$ of integer lattice (that is $A^j=e^j$, e^j — unit vectors, $j=1,\dots,n$) degrees of monomials z^x on the set of variables are defined as follows $x_1+x_2+\dots+x_n$ and degrees of the monomials are the same when their exponents lie on the hyperplane $x_1+x_2+\dots+x_n=d$. Note that $\nu=e^1+e^2+\dots+e^n$ is normal vector to this hyperplane. In the case of an arbitrary simplicial rational cone generated by the vectors $a^1,a^2,\dots,a^n$ it is natural to take $\nu=a^1+a^2+\dots+a^n$ and denote $\langle\nu,x\rangle=\nu_1x_1+\dots+\nu_nx_n$.

Theorem. If for any $\omega\in\Omega$ the condition $\langle\nu,\omega-m\rangle\leqslant 0$ is fullfield and m is the only point of Ω, which lies on the hyperplane $\langle\nu,x-m\rangle=0$, then problem (1)–(2) has the unique solution.