Solutions of Fractional Nabla Difference Equations - Existence & Uniqueness

Jagan Mohan Jonnalagadda

Department of Mathematics,
Birla Institute of Technology and Science Pilani,
Hyderabad - 500078,
Telangana, India.
j.jaganmohan@hotmail.com

The first and foremost step in qualitative study of fractional nabla difference equations is to establish sufficient conditions on existence and uniqueness of its solutions. Unlike delta difference equations, existence and uniqueness of solutions is not obvious for nabla difference equations. For example, consider a nonautonomous nabla difference equation together with an initial condition of the form

$$\big(\nabla u\big)(t) = f(t, u(t)), \quad t \in \mathbb{N}_1, \tag{1}$$
$$u(0) = c, \tag{2}$$

where $f : \mathbb{N}_0 \times \mathbb{R} \to \mathbb{R}$, $u : \mathbb{N}_0 \to \mathbb{R}$ and $c \in \mathbb{R}$. Here $\mathbb{N}_a = \{a, a+1, a+2, ...\}$, for any $a \in \mathbb{R}$. We know that u is a solution of the initial value problem (1) - (2) if and only if it has the following representation

$$u(t) = c + \sum_{s=1}^{t} f(s, u(s)), \quad t \in \mathbb{N}_0. \tag{3}$$

Since $f(t, u(t))$ is present on the right hand side of equation (3), existence of u is not trivial. It depends on the nature of f. So we impose some restrictions on f to establish existence and uniqueness of solutions of (1) - (2).
Similarly we require few conditions on $\mathbf{f}$, which we establish in this article, to obtain existence, uniqueness and dependency of solutions of initial value problems associated with nonautonomous fractional nabla difference systems of the form

$$\big(\nabla^{\alpha}_{-1}\mathbf{u}\big)(t) = \mathbf{f}(t, \mathbf{u}(t)), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{4}$$
$$\nabla^{-(1-\alpha)}_{-1}\mathbf{u}(t)\Big|_{t=0} = \mathbf{u}(0) = \mathbf{c}, \tag{5}$$

and

$$\big(\nabla^{\alpha}_{0*}\mathbf{u}\big)(t) = \mathbf{f}(t, \mathbf{u}(t)), \quad 0 < \alpha < 1, \quad t \in \mathbb{N}_1, \tag{6}$$
$$\mathbf{u}(0) = \mathbf{c}, \tag{7}$$

where ∇_{-1}^{α} and ∇_{0*}^{α} are the Riemann - Liouville and Caputo type fractional difference operators, $\mathbf{u}(t)$ is an n-vector whose components are functions of the variable t, $\mathbf{c}$ is a constant n-vector and $\mathbf{f}(t, \mathbf{u}(t))$ is an n-vector whose components are functions of the variable t and the n-vector $\mathbf{u}(t)$.
We provide few examples to illustrate the applicability of main results. We also study existence and uniqueness of solutions of logistic equation, prey - predator system and SIR epidemic system in discrete fractional nabla perspective as an application of established results in this paper..