Non-Ergodic Lotka-Volterra Operators

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Let $S^{m-1} = \{x = (x_1, \cdots, x_m) \in \mathbb{R}^m : x_i \geq 0, \sum_{i=1}^m x_i = 1\}$ be the $(m-1)$-dimensional standard simplex. A mapping $V : S^{m-1} \to S^{m-1}$, $V(x) = x'$ such that $x'_k = \sum_{i,j=1}^m p_{ijk} x_i x_j$ for all $x \in S^{m-1}$ and for all $k = \overline{1,m}$ is called a quadratic stochastic operator (in short q.s.o.), where $p_{ijk} = p_{jik} \geq 0$ and $\sum_{k=1}^m p_{ijk} = 1$ for all i, j, k. Based on some numerical calculations, S.M. Ulam conjectured [1] that the ergodic theorem holds true for any q.s.o. V, that is the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^{(k)}(x)$ exists for any $x \in S^{m-1}$ where $V^{(k+1)} = V \circ V^{(k)}$. However, M.I. Zakharevich showed [2] that Ulam's conjecture is false in general. Namely, for the q.s.o. $V : S^2 \to S^2$, $V(x) = x'$ where $x'_1 = x_1^2 + 2x_1x_2$, $x'_2 = x_2^2 + 2x_2x_3$, and $x'_3 = x_3^2 + 2x_1x_3$, the limit $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} V^{(k)}(x)$ does not exist for any $x \in \mathrm{Int}S^2 \setminus \{(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\}$.
A mapping $V : S^{m-1} \to S^{m-1}$ is called a Lotka-Volterra operator (in short LV-operator) if every face of the simplex is invariant under V, i.e., one has that $V(S_\alpha) \subset S_\alpha$ for any $\alpha \subset \{1, \cdots, m\}$ where $S_\alpha = conv\{e_k\}_{k\in\alpha}$ and $\{e_k\}_{i=1}^m$ is the standard basis of $\mathbb{R}^m$. An LV-operator is a discrete analogy of a generalized predator-prey model. It is worth of mentioning [3] that a q.s.o. is an LV-operator if and only if $p_{ijk} = 0$ whenever $k \notin \{i, j\}$. In the paper [4], a necessary condition was provided for ergodicity of q.s.o. defined on the 2D simplex. In other words, Zakharevich's result was generalized in some class of LV-q.s.o.
Let $f : S^2 \to [-1, 1]$ be any C^1-smooth functional (having the first order continuous partial derivatives). We define an LV-operator $V_f : S^2 \to S^2$, $V_f(x, y, z) = (x', y', z')$ as follows $x' = x[1 + (ay - bz)f(x, y, z)]$, $y' = y[1 + (cz - ax)f(x, y, z)]$, $z' = z[1 + (bx - cy)f(x, y, z)]$ where $a, b, c \in [-1, 1]$. It is clear that if $f \equiv const$ then the LV-operator V_f is a q.s.o. which was studied in the paper [4]. In this paper, we provide a sufficient condition in which the ergodic theorem will fail for V_f.

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