On embeddability of homeomorphisms of the circle in generalized flows

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Let $\mathbb{S}^1$ be the unite circle with positive orientation and $\mathrm{cc}[\mathbb{S}^1]$ be the family of all convex and compact subsets od $\mathbb{S}^1$. A family $\{f^t, t \in \mathbb{R}\}$ of homeomorphisms $f^t \colon \mathbb{S}^1 \to \mathbb{S}^1$ such that

$$f^t \circ f^s = f^{t+s}, \quad t, s \in \mathbb{R}$$

is said to be an *iteration group* or a *flow*. If for every $z \in \mathbb{S}^1$ the mapping $t \to f^t(z)$ is continuous then iteration group (flow) is said to be *continuous*. A homeomorphism $F \colon \mathbb{S}^1 \to \mathbb{S}^1$ is said to be *(continuously) embeddable* if there exists a (continuous) flow $\{f^t, t \in \mathbb{R}\}$ such that $f^1 = F$. This flow is said to be *the embedding of* F. A homeomorphism F without periodic points is continuously embeddable if and only if F is minimal. Here we consider the homeomorphisms which are not minimal. In such a case, under some strong assumptions, F may possess embedding. If it exists then it is very irregular (nonmeasurable with respect the time parameter), and moreover, F has infinitely many such embeddings. To eliminate this inconvenience we propose a new approach to the problem, which assures a regular embeddability for all homeomorphisms of the circle without periodic points. For a given homeomorphism $F \colon \mathbb{S}^1 \to \mathbb{S}^1$ without periodic points we construct some substitute of an iteration group, namely the unique special set-valued flow $\{F^t \colon \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in \mathbb{R}\}$ which is reasonable regular and such that $F(x) \in F^1(x)$. This set-valued flow is a single-valued if and only if F is a minimal homeomorphism. We also determine a maximal subgroup $T \subset \mathbb{R}$ such that $\{F^t \colon \mathbb{S}^1 \to \mathrm{cc}[\mathbb{S}^1], t \in T\}$ has continuous selections $\{f^t \colon \mathbb{S}^1 \to \mathbb{S}^1, t \in T\}$ which are the regular embeddings of F with time parameter restricted to subgroup T which is countable and dense in $\mathbb{R}$.