On embeddability of homeomorphisms of the circle
in generalized flows

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Let $S^1$ be the unite circle with positive orientation and $cc[S^1]$ be the family of all convex and compact subsets of $S^1$. A family $\{f^t, t \in \mathbb{R}\}$ of homeomorphisms $f^t: S^1 \to S^1$ such that $f^t \circ f^s = f^{t+s}$, $t, s \in \mathbb{R}$ is said to be an iteration group or a flow. If for every $z \in S^1$ the mapping $t \to f^t(z)$ is continuous then iteration group (flow) is said to be continuous. A homeomorphism $F: S^1 \to S^1$ is said to be (continuously) embeddable if there exists a (continuous) flow $\{f^t, t \in \mathbb{R}\}$ such that $f^1 = F$. This flow is said to be the embedding of $F$. A homeomorphism $F$ without periodic points is continuously embeddable if and only if $F$ is minimal. Here we consider the homeomorphisms which are not minimal. In such a case, under some strong assumptions, $F$ may possess embedding. If it exists then it is very irregular (nonmeasurable with respect the time parameter), and moreover, $F$ has infinitely many such embeddings. To eliminate this inconvenience we propose a new approach to the problem, which assures a regular embeddability for all homeomorphisms of the circle without periodic points. For a given homeomorphism $F: S^1 \to S^1$ without periodic points we construct some substitute of an iteration group, namely the unique special set-valued flow $\{F^t: S^1 \to cc[S^1], t \in \mathbb{R}\}$ which is reasonable regular and such that $F(x) \in F^1(x)$. This set-valued flow is a single-valued if and only if $F$ is a minimal homeomorphism. We also determine a maximal subgroup $T \subset \mathbb{R}$ such that $\{F^t: S^1 \to cc[S^1], t \in T\}$ has continuous selections $\{f^t: S^1 \to S^1, t \in T\}$ which are the regular embeddings of $F$ with time parameter restricted to subgroup $T$ which is countable and dense in $\mathbb{R}$. 