On a Fredholm property for some difference equations

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One considers a linear difference operator

$$\mathcal{D}: \ u(x) \longmapsto \sum_{-\infty}^{+\infty} a_k(x) u(x+\beta_k), \quad x \in \mathbf{R},$$

and a function $\sigma(x,\xi)$ represented by the series

$$\sigma(x,\xi) = \sum_{-\infty}^{+\infty} a_k(x) e^{i\beta_k \xi},$$

is called its symbol.
We will assume here that $a_k(x), \forall k \in \mathbf{Z}, \sigma(x,\xi)$ are continuous functions on $\mathbf{R} \times \dot{\mathbf{R}}$, i.e. $\exists \lim\limits_{\xi\to\infty} \sigma(x,\xi) \in C(\mathbf{R})$. If

$$\sigma(-\infty,\xi) = \sigma(+\infty,\xi) \equiv \sigma(\xi), \tag{1}$$

then we denote γ a closed curve in a complex plane $\mathbf{C}$, which coincides with an image of the function $\sigma(\xi)$.
Theorem. *The operator $\mathcal{D}$ has a Fredholm property and Ind $\mathcal{D}$ is vanishing in the space $L_2(\mathbf{R})$ iff $\sigma(x,\xi) \neq 0, \forall x, \xi \in \dot{\mathbf{R}}$ and the winding number for γ is equal to* 0.
May be the condition (1) doesn't hold, and for this case one has a corresponding modification.
One considers also a discrete case $x \in \mathbf{Z}$ and obtains a similar result. Some variants of discrete equations were considered in [1,2].
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[1] Vasilyev, A.V., Vasilyev, V.B., Discrete singular operators and equations in a half-space, *Azerb. J. Math.* 3(2013), 84-93.
[2] Vasilyev, A.V., Vasilyev, V.B., Discrete singular integrals in a half-space, In: *Current Trends in Analysis and its Applications. Proc. 9th ISAAC Congress, Krakow, 2013.* Birkhäuser, Basel, 2015. P. 663-670.